Solutions to Complex Variables
Exam 1

1. Show that in an arbitrary small disk $\{z : |z| < \varepsilon\}$ the function $f(z) = e^{1/z}$ takes every non-zero value infinitely often.

Solution: Let $\omega \neq 0$. If $\omega = re^{i\varphi}$, $r > 0$ $0 \leq \varphi < 2\pi$ then

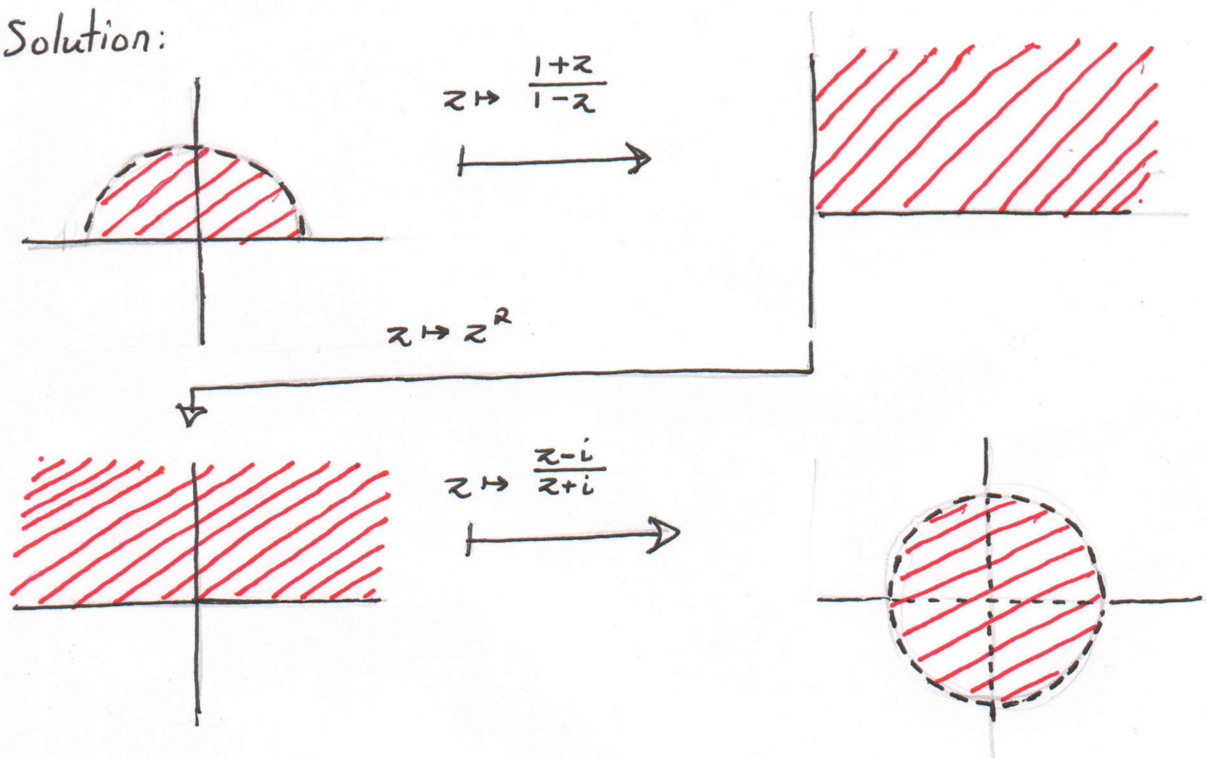
$\log \omega = \ln r + i(\varphi + 2\pi n)$ where $|\log \omega| \mapsto \infty$ as $n \mapsto \infty$.

Hence $z_n = \dfrac{1}{\ln r + i(\varphi + 2\pi n)} \mapsto 0$ as $n \mapsto \infty$. In particular,

$z_n \in \mathbb{D}_\varepsilon = \{z : |z| < \varepsilon\}$ for all but finitely many n.

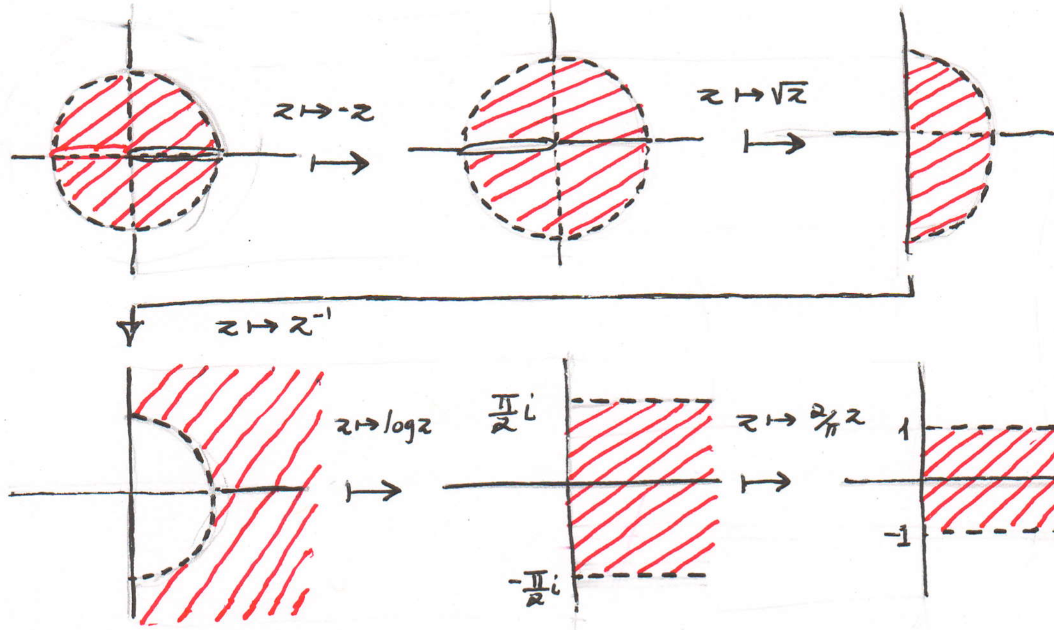
2. Find a conformal mapping from a half open unit disk onto the open unit disk.

Solution:



3. Give an explicit formula for a biholomorphism between the slit unit disk $\mathbb{D} - [0,1)$ and the half-strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 1,\ \operatorname{Re} z > 0\}$

Solution: The biholomorphism is a composition of the following functions:



4. Let Ω be a region and let $f, g : \Omega \mapsto \mathbb{C}$ be holomorphic functions satisfying $f(z)g(z) = 0$ for every $z \in \Omega$. Show that either $f \equiv 0$ or $g \equiv 0$.

Solution: Let $F = \{z \in \Omega : f(z) = 0\}$ and $G = \{z \in \Omega : g(z) = 0\}$. Since $\Omega = F \cup G$, at least one of $\{F, G\}$ is not countable. If this set happens to be F, some point $z \in F$ is a limit point of F. This implies that the zeros of f are not isolated and therefore $f \equiv 0$.

5. Show that for each $R>0$, if n is large enough,

$$P_n(z) = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}$$

has no zeros in the disk $\{z : |z| < R\}$.

Solution: Let n be large enough so that

$|e^z - P_n(z)| \leq \displaystyle\sum_{k=n+1}^{\infty} \frac{R^k}{k!} < \frac{1}{2} e^{-R}$. Then $|P_n(z)| \geq \frac{1}{2} e^{-R} > 0$

for all $z \in \mathbb{D}_R$ for if $|P_n(z)| < \frac{1}{2} e^{-R}$,

$$e^{-R} \leq |e^z| \leq |e^z - P_n(z)| + |P_n(z)| < \tfrac{1}{2} e^{-R} + \tfrac{1}{2} e^{-R} = e^{-R}$$

which is clearly a contradiction.

6. Find all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$.

Solution: Clearly $F(z) = e^z$ is one such function. If $G(z)$ is an analytic continuation of $f(x)$ to $\mathbb{C}$, then $F(z) - G(z) = 0$ for all $z \in \mathbb{R}$. This implies that $F(z) - G(z) = 0$ for all $z \in \mathbb{C}$ (Since the zeros of a non-constant holomorphic function must be isolated.)

Hence $F(z) = e^z$ is the only entire function such that $F(x) = e^x$.

7. Let f be an entire function and suppose there is a constant M, an $R_0 > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R_0$. Show that f is a polynomial of degree $\leq n$.

Solution: Pick $R > |z| > R_0$. Since f is entire,

$$f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i} \int_{C_R(z)} \frac{f(\xi)}{(\xi - z)^{n+2}} \, d\xi. \quad \text{Thus}$$

$$\left| f^{(n+1)}(z) \right| \leq \frac{(n+1)!}{2\pi} 2\pi R \sup_{0 \leq \theta \leq 2\pi} |f(z + Re^{i\theta})| \cdot R^{-n-2} \leq$$

$$\leq 2^n M R^{n+1} \cdot R^{-n-2} = \frac{2^n M}{R} \longmapsto 0 \quad \text{as } R \longmapsto \infty.$$

Therefore $f^{(n+1)}(z) = 0$. It follows that $f^{(n+1)} \equiv 0$.

Consequently f must be a polynomial with degree $\deg f \leq n$.

8. Show that an entire function f satisfying $|f(z)| \leq 1 + |z|^{1/2}$ for any z must be a constant function.

Solution: We show that $f'(z) = 0$. This will imply that $f(z) = C$

Now $$f'(z) = \frac{1}{2\pi i} \int_{C_R(z)} \frac{f(\xi)}{(\xi - z)^2} \, d\xi \quad \text{and}$$

$$|f'(z)| \leq \frac{1}{2\pi} \cdot \sup_{0 \leq \theta \leq 2\pi} \frac{1 + |z + Re^{i\theta}|^{1/2}}{R^2} 2\pi R \leq \frac{\sqrt{2R} + 1}{R} \longmapsto 0$$

as $R \longmapsto \infty$. Thus $f'(z) = 0$ as desired.

9. Show that an entire function f with $\mathrm{Re}\, f > 0$ must be constant.

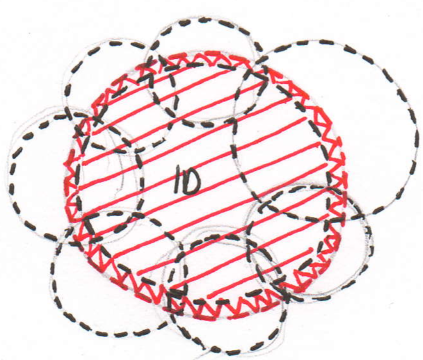
Solution: Consider $g(z) = e^{-f(z)}$. Then g is an entire function that satisfies $|g(z)| = e^{-\mathrm{Re}\, f(z)} \leq 1$. Hence $g(z)$ is constant. This means that $f(z) = c + i(\varphi + 2\pi n)$.
It remains to show that n is fixed.

Let $n(z) = \frac{1}{2\pi i}\left(f(z) - c - i\varphi\right)$ is an integer valued entire function. Since $n(\mathbb{C}) \subset \mathbb{Z}$ is connected it must be constant.

10. Suppose $f: \mathbb{D} \mapsto \mathbb{C}$ is holomorphic in the unit disk $\mathbb{D}$ and the radius of convergence of the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is 1. Show that f has at least one singular point on the boundary unit circle.

Solution: Assume that f has no singular points on $S^1 = \partial D$.

Then we can cover S^1 by finitely many disks centered on the boundary such that f is holomorphic on these disks.



The disks surrounding the boundary S^1 allow us to extend the disk $\mathbb{D}$ slightly as shown in the picture. This implies that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $1+\varepsilon > 1$. Since this contradicts our hypothesis, singularities must be present on S^1.

11. Let $f: \mathbb{D} \mapsto \mathbb{C}$ be holomorphic in the unit disk $\mathbb{D}$ and continuous on the boundary $S^1 = \partial \mathbb{D}$. Suppose there is an open arc $I \subset S^1$ such that $f_I = 0$. Show that $f = 0$ everywhere in $\mathbb{D}$.

Solution: Let $\varphi : \mathbb{H} \mapsto \mathbb{D}$ be given by $\varphi(z) = \frac{z-i}{z+i}$. where $\mathbb{H} = \{ z \in \mathbb{C} : \Im m z > 0 \}$.

Define $F: \mathbb{H} \mapsto \mathbb{H}$ by $F(z) = \varphi^{-1} \circ f \circ \varphi(z)$. Clearly $F(\mathbb{R}) \subseteq \mathbb{R}$. By the Schwarz Reflection Principle we can extend F to $\tilde{F} : \mathbb{C} \mapsto \mathbb{C}$ given by

$$\tilde{F}(z) = \begin{cases} F(z) & \text{if } \Im m z \geq 0 \\ \overline{F(\bar{z})} & \text{if } \Im m z < 0 \end{cases}$$

Since $\tilde{F}(\varphi^{-1}(I)) = \varphi^{-1} f(I) = \varphi^{-1}(0) = i$, $\tilde{F}$ must be identically equal to i everywhere on $\mathbb{C}$. $\Rightarrow F(z) = $
$= \varphi^{-1} f \varphi(z) = i \Rightarrow f(\varphi(\mathbb{H})) = f(\mathbb{D}) = \varphi(i) = 0.$